

RAKHMANOV'S THEOREM FOR ORTHOGONAL MATRIX POLYNOMIALS ON THE UNIT CIRCLE

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ABSTRACT. Rakhmanov's theorem for orthogonal polynomials on the unit circle gives a sufficient condition on the orthogonality measure for orthogonal polynomials on the unit circle, in order that the reflection coefficients (the recurrence coefficients in the Szegő recurrence relation) converge to zero. In this paper we give the analog for orthogonal matrix polynomials on the unit circle.

1. RAKHMANOV'S THEOREM IN THE SCALAR CASE

Let $\varphi_n(z) = \kappa_n z^n + \dots$ ($n = 0, 1, 2, \dots$), with $\kappa_n > 0$, be orthonormal polynomials on the unit circle with respect to some positive measure μ :

$$\int_0^{2\pi} \varphi_n(z) \overline{\varphi_m(z)} d\mu(\theta) = \delta_{m,n}, \quad z = e^{i\theta}.$$

We denote the monic polynomials by $\Phi_n(z) = \varphi_n(z)/\kappa_n$. These monic polynomials satisfy a useful recurrence relation

$$(1.1) \quad \Phi_n(z) = z\Phi_{n-1}(z) + \Phi_n(0)\Phi_{n-1}^*(z),$$

where $\Phi_n^*(z) = z^n \overline{\Phi_n(1/z)}$ is the reversed polynomial (see, e.g., [12,15]). The coefficients $\Phi_n(0)$, which act as recurrence coefficients in this recurrence relation, are known as reflection coefficients and $\alpha_n = -\overline{\Phi_{n+1}(0)}$ are called Verblunsky coefficients in [12]. It is well known that all the zeros $z_{k,n}$ of φ_n lie in the open unit disk, and hence $|\Phi_n(0)| = \prod_{k=1}^n |z_{k,n}| < 1$. Moreover,

$$(1.2) \quad \frac{\kappa_{n-1}^2}{\kappa_n^2} = 1 - |\Phi_n(0)|^2,$$

so that the reflection coefficients allow us to compute the monic orthogonal polynomials recursively using (1.1), but also the orthonormal polynomials using in addition

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(1.2). Conversely, given a sequence a_n of complex numbers for which $|a_n| < 1$ for all $n > 0$, then polynomials satisfying

$$\Phi_n(z) = z\Phi_{n-1}(z) + a_n\Phi_{n-1}^*(z),$$

with $\Phi_0 = 1$, will be monic orthogonal polynomials on the unit circle for a unique orthogonality measure μ , and the reflection coefficients of these polynomials are $\Phi_n(0) = a_n$ (Favard's theorem for orthogonal polynomials on the unit circle or Geronimus's theorem, see, e.g., [3] and [12]).

It is straightforward to see that the condition

$$(1.3) \quad \lim_{n \rightarrow \infty} \Phi_n(0) = 0$$

implies that

$$(1.4) \quad \lim_{n \rightarrow \infty} \frac{\kappa_{n-1}^2}{\kappa_n^2} = 1$$

and from (1.1) we also see that

$$(1.5) \quad \lim_{n \rightarrow \infty} \frac{\Phi_n(z)}{\Phi_{n-1}(z)} = z$$

uniformly for z on the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, because on the unit circle we have $|\Phi_n^*(z)/\Phi_n(z)| = 1$. Combined with the ratio behavior (1.4) this also gives ratio behavior of the orthonormal polynomials

$$(1.6) \quad \lim_{n \rightarrow \infty} \frac{\varphi_n(z)}{\varphi_{n-1}(z)} = z, \quad z \in \mathbb{T}.$$

This indicates that the orthonormal polynomials $\varphi_n(z)$ behaves very much like the polynomials z^n , which are the orthonormal polynomials on the unit circle for Lebesgue measure $d\theta/2\pi$. The condition (1.3) is therefore a very natural condition for various asymptotic properties of the orthogonal polynomials. For this reason it is of interest to find conditions on the orthogonality measure μ that imply (1.3). Rakhmanov [11] (see also [8]) proved that a rather mild condition on the size of μ on the unit circle is sufficient.

Rakhmanov's Theorem. *Suppose that $\mu' > 0$ almost everywhere on \mathbb{T} . Then $\lim_{n \rightarrow \infty} \Phi_n(0) = 0$.*

This condition is not necessary: we know examples of discrete measures and singularly continuous measures, hence measure without absolutely continuous component, for which (1.3) holds [6], [7], [16], [14]. A nice proof of Rakhmanov's theorem uses the following two equivalences (Nevai [9] [10], Li and Saff [5])

$$(1.7) \quad \lim_{n \rightarrow \infty} \Phi_n(0) = 0 \iff \lim_{n \rightarrow \infty} \inf_{\ell \geq 1} \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{|\varphi_n(z)|^2}{|\varphi_{n+\ell}(z)|^2} - 1 \right| d\theta = 0,$$

and

$$(1.8) \quad \mu'(\theta) > 0 \text{ a.e. on } [0, 2\pi) \iff \lim_{n \rightarrow \infty} \sup_{\ell \geq 1} \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{|\varphi_n(z)|^2}{|\varphi_{n+\ell}(z)|^2} - 1 \right| d\theta = 0.$$

In this paper we will investigate the analog of Rakhmanov's theorem for orthogonal matrix polynomials on the unit circle. In Section 2 we will introduce the necessary background on orthogonal matrix polynomials and present the matrix analogs of the Szegő recurrence (1.1) and the reflection coefficients, which turn out to be matrices. In Section 3 we prove the matrix analog of the characterization (1.7) and in Section 4 we deal with the characterization (1.8).

2. ORTHOGONAL MATRIX POLYNOMIALS ON THE UNIT CIRCLE

Let ρ be a $p \times p$ Hermitian matrix-valued measure on the unit circle \mathbb{T} , then ρ induces two matrix inner products. The left inner product

$$\langle P, Q \rangle_L = \int_0^{2\pi} P(z) d\rho(\theta) Q(z)^*, \quad z = e^{i\theta},$$

where from now on Q^* is the Hermitian transpose of the matrix Q , i.e., $Q^* = \overline{Q^T}$, and the right inner product

$$\langle P, Q \rangle_R = \int_0^{2\pi} P(z)^* d\rho(\theta) Q(z), \quad z = e^{i\theta}.$$

Both inner products give rise to a sequence of orthonormal polynomials, which we call left orthonormal polynomials φ_n^L and right orthonormal polynomials φ_n^R respectively:

$$\int_0^{2\pi} \varphi_n^L(z) d\rho(\theta) [\varphi_m^L(z)]^* = \delta_{m,n} = \int_0^{2\pi} [\varphi_n^R(z)]^* d\rho(\theta) \varphi_m^R(z).$$

These orthonormal polynomials are determined up to multiplication by a unitary matrix (on the left for the left orthonormal polynomials and on the right for the right orthonormal polynomials). For background and general results on these orthogonal polynomials on the unit circle, we refer to [1] and [13].

We use the notation $\tilde{P}(z) = z^n P(1/\bar{z})^*$ for the reversed matrix polynomial, whenever P is a polynomial of degree at most n . If we write the orthonormal polynomials as

$$\begin{aligned} \varphi_n^L(z) &= L_{n,n} z^n + L_{n,n-1} z^{n-1} + \cdots + L_{n,0} \\ \varphi_n^R(z) &= K_{n,n} z^n + K_{n,n-1} z^{n-1} + \cdots + K_{n,0} \end{aligned}$$

then the leading coefficients $K_{n,n}$ and $L_{n,n}$ are non-singular $p \times p$ matrices. The identity $\langle \varphi_n^R, \tilde{\varphi}_n^L \rangle_R = \langle \tilde{\varphi}_n^R, \varphi_n^L \rangle_L$ implies

$$L_{n,0}^* L_{n,n} = K_{n,n} K_{n,0}^*,$$

and we can then introduce the reflection coefficients as

$$(2.1) \quad H_n = (L_{n,n}^*)^{-1} K_{n,0} = L_{n,0} (K_{n,n}^*)^{-1}.$$

One can show that $I - H_n H_n^*$ and $I - H_n^* H_n$ are positive definite, so that $\|H_n\|_2 < 1$ for all $n > 0$, and

$$\begin{aligned} (I - H_n^* H_n)^{1/2} &= K_{n,n}^{-1} K_{n-1,n-1} \\ (I - H_n H_n^*)^{1/2} &= (L_{n,n}^*)^{-1} L_{n-1,n-1}^*, \end{aligned}$$

which is the matrix analog of (1.2), and furthermore we have the recurrences

$$(2.2) \quad (I - H_n H_n^*)^{1/2} \varphi_n^L(z) = z \varphi_{n-1}^L(z) + H_n \tilde{\varphi}_{n-1}^R(z)$$

$$(2.2) \quad \varphi_n^R(z) (I - H_n^* H_n)^{1/2} = z \varphi_{n-1}^R(z) + \tilde{\varphi}_{n-1}^L(z) H_n$$

which are the matrix analogs of the Szegő recurrences (1.1), but for orthonormal polynomials. Observe that for orthogonal matrix polynomials on the unit circle we need to work with both left and right orthogonal polynomials. Again there is a converse result (Favard's theorem for orthogonal matrix polynomials on the unit circle) saying that matrix polynomials φ_n^L and φ_n^R satisfying the recurrences (2.2)–(2.3), with initial conditions φ_0^L and φ_0^R which are non-singular matrices with $[\varphi_0^L]^* \varphi_0^L = \varphi_0^R [\varphi_0^R]^*$ and with matrix coefficients H_n for which $\|H_n\|_2 < 1$ for all $n > 0$, are always orthonormal matrix polynomials on the unit circle for some positive matrix measure ρ [1, Thm. 15 on p. 155].

Using (2.2) and (2.3) and the identity $(I - AA^*)^{-1}A = A(1 - A^*A)^{-1}$, we can find

$$\tilde{\varphi}_n^L(z) \varphi_n^L(\xi) - \xi \tilde{\varphi}_{n-1}^L(z) \varphi_{n-1}^L(\xi) = \varphi_n^R(z) \tilde{\varphi}_n^R(z) - z \varphi_{n-1}^R(z) \tilde{\varphi}_{n-1}^R(\xi).$$

Multiply both sides by z^{-n} , then summing over n gives

$$z^{-n} \varphi_n^R(z) \tilde{\varphi}_n^R(\xi) = (1 - \xi/z) \sum_{k=0}^n z^{-k} \tilde{\varphi}_k^L(z) \varphi_k^L(\xi) + \xi z^{-n} \tilde{\varphi}_n^L(z) \varphi_n^L(\xi).$$

Now use $\tilde{\varphi}_n^L(z) = z^n \varphi_n^L(1/\bar{z})^*$ and replace z by $1/\bar{z}$, then this gives the *Christoffel-Darboux formula*

$$(2.4) \quad (1 - \xi \bar{z}) \sum_{k=0}^n [\varphi_k^L(z)]^* \varphi_k^L(\xi) = [\tilde{\varphi}_n^R(z)]^* \tilde{\varphi}_n^R(\xi) - \xi \bar{z} [\varphi_n^L(z)]^* \varphi_n^L(\xi).$$

In a similar way we can obtain the dual formula

$$(2.5) \quad (1 - \xi \bar{z}) \sum_{k=0}^n \varphi_k^R(\xi) [\varphi_k^R(z)]^* = \tilde{\varphi}_n^L(\xi) [\tilde{\varphi}_n^L(z)]^* - \xi \bar{z} \varphi_n^R(\xi) [\varphi_n^R(z)]^*.$$

An important consequence of this Christoffel-Darboux formula is when we take $z = \xi$ on the unit circle, which gives the identity

$$(2.6) \quad \varphi_n^R(z) \tilde{\varphi}_n^R(z) = \tilde{\varphi}_n^L(z) \varphi_n^L(z), \quad z = e^{i\theta}.$$

Observe that on the unit circle $\tilde{P}(z) = z^n P(z)^*$ so that $z^{-n} \tilde{\varphi}_n^L(z) \varphi_n^L(z)$ is a positive definite matrix for every z on the unit circle. Of particular use is the absolutely continuous matrix measure

$$(2.7) \quad d\rho_n(\theta) = ([\varphi_n^L(z)]^* \varphi_n^L(z))^{-1} d\theta/2\pi = (\varphi_n^R(z) [\varphi_n^R(z)]^*)^{-1} d\theta/2\pi, \quad z = e^{i\theta},$$

because the first $n+1$ orthogonal matrix polynomials are also orthogonal with respect to this measure ([1, Eq. (71) on p. 155])

$$(2.8) \quad \frac{1}{2\pi} \int_0^{2\pi} \varphi_k^L(z) ([\varphi_n^L(z)]^* \varphi_n^L(z))^{-1} [\varphi_m^L(z)]^* d\theta = \delta_{k,m}, \quad k, m \leq n,$$

and similarly for the right orthogonal matrix polynomials.

We are now ready to state Rakhmanov's theorem for orthogonal matrix polynomials on the unit circle.

Theorem. *Suppose that ρ is a matrix measure on the unit circle with $d\rho(\theta) = \rho'(\theta) d\theta/2\pi + d\rho_s(\theta)$ and ρ_s the singular part of the measure. If $\det \rho'(\theta) > 0$ almost everywhere on $[0, 2\pi)$ then $\lim_{n \rightarrow \infty} H_n = 0$.*

We will prove this using appropriate matrix analogs of (1.7) (see section 3) and (1.8) (see section 4).

3. CHARACTERIZATION OF $\lim_{n \rightarrow \infty} H_n = 0$

Lemma 1. *Let $\varphi_n^L(z)$ and $\varphi_n^R(z)$ be left and right orthonormal matrix polynomials with reflection coefficients H_n , then*

$$(3.1) \quad \lim_{n \rightarrow \infty} H_n = 0 \iff \lim_{n \rightarrow \infty} \inf_{\ell \geq 1} \frac{1}{2\pi} \int_0^{2\pi} \|\varphi_n^L(z) \varphi_{n+\ell}^L(z)^{-1} [\varphi_{n+\ell}^L(z)^*]^{-1} \varphi_n^L(z)^* - I\|_2 d\theta = 0.$$

Proof of \Rightarrow . The recurrence relation (2.2) gives

$$(3.2) \quad (I - H_n H_n^*)^{1/2} \varphi_n^L(z) \varphi_{n-1}^L(z)^{-1} = zI + H_n \tilde{\varphi}_{n-1}^R(z) \varphi_{n-1}^L(z)^{-1},$$

and similarly (2.3) gives

$$(3.3) \quad \varphi_{n-1}^R(z)^{-1} \varphi_n^R(z) (I - H_n^* H_n)^{1/2} = zI + \varphi_{n-1}^R(z)^{-1} \tilde{\varphi}_{n-1}^L(z) H_n.$$

Observe that $\tilde{\varphi}_{n-1}^R(z) \varphi_{n-1}^L(z)^{-1} = \varphi_{n-1}^R(z)^{-1} \tilde{\varphi}_{n-1}^L(z)$ whenever z is on the unit circle, which follows from (2.6). Furthermore this rational matrix function is a unitary matrix when z is on the unit circle. To see this, we examine the product

$$[\tilde{\varphi}_{n-1}^R(z) \varphi_{n-1}^L(z)^{-1}] [\tilde{\varphi}_{n-1}^R(z) \varphi_{n-1}^L(z)^{-1}]^*,$$

which is equal to

$$\tilde{\varphi}_{n-1}^R(z) [\varphi_{n-1}^L(z)^* \varphi_{n-1}^L(z)]^{-1} \tilde{\varphi}_{n-1}^R(z)^*.$$

Use (2.6) to replace the product of the two matrix polynomials, then we find that this expression is

$$\tilde{\varphi}_{n-1}^R(z) [\varphi_{n-1}^R(z) \varphi_{n-1}^R(z)^*]^{-1} \tilde{\varphi}_{n-1}^R(z)^*$$

and this is

$$\tilde{\varphi}_{n-1}^R(z) [\varphi_{n-1}^R(z)^*]^{-1} \varphi_{n-1}^R(z)^{-1} \tilde{\varphi}_{n-1}^R(z)^*.$$

On the unit circle we have $\tilde{\varphi}_{n-1}^R(z) = z^{n-1} \varphi_{n-1}^R(z)^*$, hence the expression reduces to the unit matrix, so that

$$[\tilde{\varphi}_{n-1}^R(z) \varphi_{n-1}^L(z)^{-1}] [\tilde{\varphi}_{n-1}^R(z) \varphi_{n-1}^L(z)^{-1}]^* = I$$

and $\tilde{\varphi}_{n-1}^R(z) \varphi_{n-1}^L(z)^{-1}$ is unitary. Consequently for the spectral norm we have $\|\tilde{\varphi}_{n-1}^R(z) \varphi_{n-1}^L(z)^{-1}\|_2 = 1$. Returning to (3.2), we now have

$$\|(I - H_n H_n^*)^{1/2} \varphi_n^L(z) \varphi_{n-1}^L(z)^{-1} - zI\|_2 \leq \|H_n\|_2$$

and similarly (3.3) gives

$$\|\varphi_{n-1}^R(z)^{-1} \varphi_n^R(z) (I - H_n^* H_n)^{1/2} - zI\|_2 \leq \|H_n\|_2.$$

Hence $\lim_{n \rightarrow \infty} H_n = 0$ implies

$$\lim_{n \rightarrow \infty} \varphi_n^L(z) \varphi_{n-1}^L(z)^{-1} = zI, \quad \lim_{n \rightarrow \infty} \varphi_{n-1}^R(z)^{-1} \varphi_n^R(z) = zI.$$

uniformly for $z \in \mathbb{T}$. This implies that for $\ell \geq 1$ fixed

$$\lim_{n \rightarrow \infty} \varphi_n^L(z) \varphi_{n+\ell}^L(z)^{-1} = z^{-\ell} I,$$

uniformly on the unit circle, which implies that

$$\lim_{n \rightarrow \infty} \inf_{\ell \geq 1} \frac{1}{2\pi} \int_0^{2\pi} \|\varphi_n^L(z) \varphi_{n+\ell}^L(z)^{-1} [\varphi_{n+\ell}^L(z)^*]^{-1} \varphi_n^L(z)^* - I\|_2 d\theta = 0,$$

which is what we wanted to prove. Observe that we also get a similar result for the right orthogonal polynomials.

Proof of \Leftarrow . Here we rely on the identity

$$(3.4) \quad \langle z\varphi_n^L, \tilde{\varphi}_n^R \rangle_L = -H_{n+1}.$$

Indeed, if we use (2.2) (replacing n by $n+1$ everywhere), then

$$\begin{aligned} \langle z\varphi_n^L, \tilde{\varphi}_n^R \rangle_L &= \int_0^{2\pi} z\varphi_n^L(z) d\rho(\theta) [\tilde{\varphi}_n^R(z)]^* \\ &= (I - H_{n+1}H_{n+1}^*)^{1/2} \langle \varphi_{n+1}^L, \tilde{\varphi}_n^R \rangle_L - H_{n+1} \langle \tilde{\varphi}_n^R, \tilde{\varphi}_n^R \rangle_L \\ &= -H_{n+1} \langle \tilde{\varphi}_n^R, \tilde{\varphi}_n^R \rangle_L \end{aligned}$$

where the last step follows from the orthogonality. Now $\langle \tilde{P}_n, \tilde{Q}_n \rangle_L = \langle P_n, Q_n \rangle_R$, hence

$$\langle z\varphi_n^L, \tilde{\varphi}_n^R \rangle_L = -H_{n+1} \langle \varphi_n^R, \varphi_n^R \rangle_R = -H_{n+1}$$

where we used the orthonormality. This shows that (3.4) indeed holds.

From the finite orthogonality (2.8) we can easily deduce the following result for the measures ρ and ρ_n , given by (2.7):

$$\int_0^{2\pi} P_k(z) d\rho(\theta) [Q_m(z)]^* = \frac{1}{2\pi} \int_0^{2\pi} P_k(z) ([\varphi_n^L(z)]^* \varphi_n^L(z))^{-1} Q_m(z) d\theta,$$

for all matrix polynomials P_k and Q_m of degree $k \leq n$ and $m \leq n$ respectively, by expanding P_k and Q_m in a Fourier series using the left orthonormal polynomials. Since $z\varphi_n^L(z)$ is a matrix polynomial of degree $n+1$ and $\tilde{\varphi}_n^R(z)$ is of degree n , we therefore have from (3.4)

$$-H_{n+1} = \frac{1}{2\pi} \int_0^{2\pi} z\varphi_n^L(z) ([\varphi_{n+\ell}^L(z)]^* \varphi_{n+\ell}^L(z))^{-1} [\tilde{\varphi}_n^R(z)]^* d\theta,$$

for every $\ell \geq 1$. We can write this integral as

$$\begin{aligned} &\frac{1}{2\pi} \int_0^{2\pi} \varphi_n^L(z) \varphi_{n+\ell}^L(z)^{-1} [\varphi_{n+\ell}^L(z)]^*^{-1} \varphi_n^L(z)^* z[\varphi_n^L(z)]^*^{-1} \tilde{\varphi}_n^R(z)]^* d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} [\varphi_n^L(z) \varphi_{n+\ell}^L(z)^{-1}] [\varphi_n^L(z) \varphi_{n+\ell}^L(z)^{-1}]^* z\tilde{\varphi}_n^R(z)^{-1} \varphi_n^R(z) d\theta. \end{aligned}$$

By the calculus of residues we also have

$$\frac{1}{2\pi} \int_0^{2\pi} z \tilde{\varphi}_n^L(z)^{-1} \varphi_n^R(z) d\theta = 0,$$

hence we get the formula

$$-H_{n+1} = \frac{1}{2\pi} \int_0^{2\pi} ([\varphi_n^L(z) \varphi_{n+\ell}^L(z)^{-1}] [\varphi_n^L(z) \varphi_{n+\ell}^L(z)^{-1}]^* - I) z \tilde{\varphi}_n^L(z)^{-1} \varphi_n^R(z) d\theta.$$

Recall that $\tilde{\varphi}_n^L(z)^{-1} \varphi_n^R(z)$ is unitary whenever $z \in \mathbb{T}$, so that by taking the spectral norm we get

$$\|H_{n+1}\|_2 \leq \frac{1}{2\pi} \int_0^{2\pi} \|[\varphi_n^L(z) \varphi_{n+\ell}^L(z)^{-1}] [\varphi_n^L(z) \varphi_{n+\ell}^L(z)^{-1}]^* - I\|_2 d\theta.$$

Hence if

$$\lim_{n \rightarrow \infty} \inf_{\ell \geq 1} \frac{1}{2\pi} \int_0^{2\pi} \|[\varphi_n^L(z) \varphi_{n+\ell}^L(z)^{-1}] [\varphi_n^L(z) \varphi_{n+\ell}^L(z)^{-1}]^* - I\|_2 d\theta = 0,$$

the obviously $H_n \rightarrow 0$, which is what we wanted to prove. \square

4. CHARACTERIZATION OF $\det \rho'(\theta) > 0$ ALMOST EVERYWHERE

In this section we will only use left orthogonal polynomials φ_n^L and to simplify the notation we therefore will drop the superscript L .

In general the matrix measure ρ will consist of an absolutely continuous part with Radon-Nikodym derivative ρ' , and a singular part ρ_s . A remarkable fact is that when $\det \rho'(\theta) > 0$ almost everywhere, then the singular part ρ_s does not interfere in the ratio asymptotic behavior. We can indeed annihilate the singular part using the same ideas as in [10].

Lemma 2. *Suppose ρ_s is a positive definite matrix measure on the unit circle which is singular with respect to the Lebesgue matrix measure. Then there exists a sequence of matrix functions G_n on the unit circle, such that $G_n G_n^* \leq I$,*

$$\lim_{n \rightarrow \infty} G_n G_n^* = I, \quad \text{almost everywhere on } \mathbb{T},$$

and

$$\lim_{n \rightarrow \infty} \int_0^{2\pi} G_n(\theta) d\rho_s(\theta) G_n^*(\theta) = 0.$$

Proof. From Lemma 5 in [10] we know that if μ_s is a (scalar) singular measure on the unit circle, then there is a sequence of real-valued 2π -periodic continuous functions h_n on the unit circle, such that $0 \leq h_n(\theta) \leq 1$ for all $\theta \in [0, 2\pi)$, with

$$\lim_{n \rightarrow \infty} h_n(\theta) = 1, \quad \text{almost everywhere}$$

and

$$\lim_{n \rightarrow \infty} \int_0^{2\pi} h_n(\theta) d\mu_s(\theta) = 0.$$

Let

$$\rho_s(\theta) = V(\theta) \begin{pmatrix} \rho_{s,1}(\theta) & & & 0 \\ & \rho_{s,2}(\theta) & & \\ & & \ddots & \\ 0 & & & \rho_{s,p}(\theta) \end{pmatrix} V^*(\theta)$$

be the Schur decomposition of ρ_s , with V a unitary matrix function. Each eigenvalue $\rho_{s,i}$ is singular with respect to Lebesgue measure, hence there exists a sequence $h_{n,i}$ of 2π -periodic continuous functions such that $0 \leq h_{n,i}(\theta) \leq 1$, with

$$\lim_{n \rightarrow \infty} h_{n,i}(\theta) = 1 \quad \text{almost everywhere}$$

and

$$\lim_{n \rightarrow \infty} \int_0^{2\pi} h_{n,i}(\theta) d\rho_{s,i}(\theta) = 0.$$

Consider the sequence of matrix functions

$$G_n = \begin{pmatrix} h_{n,1}^{1/2} & & & 0 \\ & h_{n,2}^{1/2} & & \\ & & \ddots & \\ 0 & & & h_{n,p}^{1/2} \end{pmatrix} V^*(\theta),$$

then $G_n G_n^* \leq I$,

$$\lim_{n \rightarrow \infty} G_n G_n^* = I, \quad \text{almost everywhere}$$

and

$$\lim_{n \rightarrow \infty} \int_0^{2\pi} G_n d\rho_s G_n^* = 0,$$

which is what we wanted to proof. \square

Lemma 3. *Let φ_n be the left orthonormal matrix polynomials with reflection coefficients H_n , then*

$$\det \rho'(x) > 0 \text{ almost everywhere} \iff$$

$$\lim_{n \rightarrow \infty} \sup_{\ell \geq 1} \frac{1}{2\pi} \int_0^{2\pi} \|\varphi_n(z) \varphi_{n+\ell}(z)^{-1} [\varphi_{n+\ell}(z)^*]^{-1} \varphi_n(z)^* - I\|_2 d\theta = 0.$$

Proof of \Leftarrow . First of all we observe that

$$\frac{1}{2\pi} \int_0^{2\pi} P_m(z) (\varphi_n(z)^* \varphi_n(z))^{-1} Q_m(z)^* d\theta = \int_0^{2\pi} P_m(\theta) d\rho(\theta) Q_m(\theta)^*$$

holds for all matrix polynomials P_m and Q_m of degree $m \leq n$. This follows from (2.8). This means that

$$\begin{aligned} & \int_0^{2\pi} F_m(\theta) \varphi_n(z) \rho'(\theta) \varphi_n(z)^* G_m(\theta)^* d\theta + \int_0^{2\pi} F_m(\theta) \varphi_n(z) d\rho_s(\theta) \varphi_n(z)^* G_m(\theta)^* \\ &= \frac{1}{2\pi} \int_0^{2\pi} F_m(\theta) \varphi_n(z) (\varphi_{n+\ell}(z)^* \varphi_{n+\ell}(z))^{-1} \varphi_n(z)^* G_m(\theta)^* d\theta, \quad \ell \geq 2m, \end{aligned}$$

for all trigonometric matrix polynomials F_m and G_m of degree at most m . This gives

$$\begin{aligned} & \frac{1}{2\pi} \int_0^{2\pi} F_m(\theta) [\varphi_n(z) 2\pi\rho'(\theta) \varphi_n(z)^* - I] G_m(\theta)^* d\theta \\ &= - \int_0^{2\pi} F_m(\theta) \varphi_n(z) d\rho_s(\theta) \varphi_n(z)^* G_m(\theta)^* \\ &+ \frac{1}{2\pi} \int_0^{2\pi} F_m(\theta) \left[\varphi_n(z) (\varphi_{n+\ell}(z)^* \varphi_{n+\ell}(z))^{-1} \varphi_n(z)^* - I \right] G_m(\theta)^* d\theta, \end{aligned}$$

whenever $\ell \geq 2m$, and if we take the spectral norm, then

$$\begin{aligned} & \left\| \frac{1}{2\pi} \int_0^{2\pi} F_m(\theta) [\varphi_n(z) 2\pi\rho'(\theta) \varphi_n(z)^* - I] G_m(\theta)^* d\theta \right\|_2 \\ & \leq \left\| \int_0^{2\pi} F_m(\theta) \varphi_n(z) d\rho_s(\theta) \varphi_n(z)^* G_m(\theta)^* \right\|_2 \\ & \quad + \max_{\theta \in [0, 2\pi)} \|F_m(\theta)\|_2 \max_{\theta \in [0, 2\pi)} \|G_m(\theta)\|_2 \\ & \quad \times \sup_{\ell \geq 1} \frac{1}{2\pi} \int_0^{2\pi} \|\varphi_n(z) (\varphi_{n+\ell}(z)^* \varphi_{n+\ell}(z))^{-1} \varphi_n(z)^* - I\|_2 d\theta, \end{aligned}$$

for all trigonometric matrix polynomials P_m and Q_m . Every 2π -periodic continuous matrix function can be uniformly approximated by trigonometric matrix polynomials. Therefore we also have

$$\begin{aligned} (3.8) \quad & \left\| \frac{1}{2\pi} \int_0^{2\pi} F(\theta) [\varphi_n(z) 2\pi\rho'(\theta) \varphi_n(z)^* - I] G(\theta)^* d\theta \right\|_2 \\ & \leq \left\| \int_0^{2\pi} F(\theta) \varphi_n(z) d\rho_s(\theta) \varphi_n(z)^* G(\theta)^* \right\|_2 \\ & \quad + \max_{\theta \in [0, 2\pi)} \|F(\theta)\|_2 \max_{\theta \in [0, 2\pi)} \|G(\theta)\|_2 \\ & \quad \times \sup_{\ell \geq 1} \frac{1}{2\pi} \int_0^{2\pi} \|\varphi_n(z) (\varphi_{n+\ell}(z)^* \varphi_{n+\ell}(z))^{-1} \varphi_n(z)^* - I\|_2 d\theta, \end{aligned}$$

for all 2π -periodic continuous matrix functions F and G . Furthermore, every matrix function F for which $\sup_{\theta \in [0, 2\pi)} \|F(\theta)\|_2 < \infty$ (i.e., $F \in L^\infty[0, 2\pi]$) can be approximated pointwise by 2π -periodic continuous matrix functions F_k with $\sup_{\theta \in [0, 2\pi)} \|F_k(\theta)\|_2 = \sup_{\theta \in [0, 2\pi)} \|F(\theta)\|_2$, hence (3.8) also holds for matrix functions $F, G \in L^\infty[0, 2\pi]$. A useful choice is to take $F = HP$ and $G = P$, where P is a unitary matrix function such that $P(\theta) \varphi_n(z) 2\pi\rho'(\theta) \varphi_n(z)^* P(\theta)^*$ is a diagonal matrix D_n containing the eigenvalues $d_{1,n}(\theta), \dots, d_{p,n}(\theta)$ of the positive definite matrix $\varphi_n(z) 2\pi\rho'(\theta) \varphi_n(z)^*$, and H is the unitary diagonal matrix with entries $\text{sign}(d_{1,n} - 1), \text{sign}(d_{2,n} - 1), \dots, \text{sign}(d_{p,n} - 1)$. Then we get

$$\begin{aligned} (3.9) \quad & \max_{1 \leq k \leq p} \int_0^{2\pi} |d_{k,n}(\theta) - 1| d\theta \\ & \leq \sup \int_0^{2\pi} \|\varphi_n(z) (\varphi_{n+\ell}(z)^* \varphi_{n+\ell}(z))^{-1} \varphi_n(z)^* - I\|_2 d\theta. \end{aligned}$$

Observe that $\det \varphi_n(z) \rho'(\theta) \varphi_n(z)^* = [\det \varphi_n(z)]^2 \det \rho'(\theta)$, and since φ_n has no zeros on the unit circle, it follows that $\rho'(\theta) = 0$ if and only $d_{k,n}(\theta) = 0$ for some $k = 1, 2, \dots, p$. Hence

$$A := \{\theta \in [0, 2\pi) : \rho'(\theta) = 0\} = \bigcup_{k=1}^p \{\theta \in [0, 2\pi) : d_{k,n}(\theta) = 0\} := \bigcup_{k=1}^p A_k.$$

Let m be Lebesgue measure on $[0, 2\pi)$, i.e., $dm(\theta) = d\theta$, then

$$m(A) = \int_A 1 d\theta = \int_{\bigcup A_k} 1 d\theta \leq \sum_{k=1}^p \int_{A_k} 1 d\theta.$$

Now, on A_k we have $d_{k,n} = 0$, hence

$$m(A) \leq \sum_{k=1}^p \int_{A_k} |d_{k,n}(\theta) - 1| d\theta \leq \sum_{k=1}^p \int_0^{2\pi} |d_{k,n}(\theta) - 1| d\theta.$$

Now use (3.9) to conclude

$$m(A) \leq p \sup_{\ell \geq 1} \int_0^{2\pi} \|\varphi_n(z) (\varphi_{n+\ell}(z)^* \varphi_{n+\ell}(z))^{-1} \varphi_n(z)^* - I\|_2 d\theta.$$

Hence if the right hand side tends to zero, then $\det \rho' > 0$ almost everywhere on the unit circle. \square

In order to prove the necessary part of the lemma, we will need some inequalities for the trace of matrices. We will always be using $p \times p$ matrices and for a matrix A we will denote its eigenvalues by $\lambda_i(A)$ ($i = 1, 2, \dots, p$) and its singular values by $\sigma_i(A)$ ($i = 1, 2, \dots, p$). Recall that for a matrix A we always have $\sigma_i(A) = \lambda_i^{1/2}(AA^*)$ and for a positive definite matrix A we have $\sigma_i(A) = \lambda_i(A)$. The trace $\text{Tr}(A)$ of a matrix A is given by $\text{Tr}(A) = \sum_{i=1}^p \lambda_i(A)$.

We will often be using the inequality

$$(5.1) \quad \sum_{i=1}^p \sigma_i^q(AB) \leq \sum_{i=1}^p \sigma_i^q(A) \sigma_i^q(B),$$

(see, e.g., Thm. 3.3.14 on p. 176 in [4]), and certainly we have for a positive definite matrix

$$(5.2) \quad \|A\|_2 = \max_{1 \leq i \leq p} \lambda_i(A) \leq \text{Tr}(A).$$

Proof of \Rightarrow . The proof consists of a few steps, each of which gives an estimate of an integral.

First we start with

$$\begin{aligned} & \int_0^{2\pi} \|\varphi_n(z) \varphi_{n+\ell}^{-1}(z) [\varphi_{n+\ell}^*(z)]^{-1} \varphi_n^*(z) - I\|_2 d\theta \\ & \leq \int_0^{2\pi} \|(\varphi_n(z) \varphi_{n+\ell}^{-1}(z) [\varphi_{n+\ell}^*(z)]^{-1} \varphi_n^*(z))^{1/2} - I\|_2 \\ & \quad \times \|(\varphi_n(z) \varphi_{n+\ell}^{-1}(z) [\varphi_{n+\ell}^*(z)]^{-1} \varphi_n^*(z))^{1/2} + I\|_2 d\theta \\ & \leq \left(\int_0^{2\pi} \|(\varphi_n(z) \varphi_{n+\ell}^{-1}(z) [\varphi_{n+\ell}^*(z)]^{-1} \varphi_n^*(z))^{1/2} - I\|_2^2 d\theta \right. \\ & \quad \left. \times \int_0^{2\pi} \|(\varphi_n(z) \varphi_{n+\ell}^{-1}(z) [\varphi_{n+\ell}^*(z)]^{-1} \varphi_n^*(z))^{1/2} + I\|_2^2 d\theta \right)^{1/2}, \end{aligned}$$

where we have used Cauchy-Schwarz for the last inequality. For the second integral on the right we have by (5.2)

$$\begin{aligned}
 (5.3) \quad & \int_0^{2\pi} \|(\varphi_n(z)\varphi_{n+\ell}^{-1}(z)[\varphi_{n+\ell}^*(z)]^{-1}\varphi_n^*(z))^{1/2} + I\|_2^2 d\theta \\
 & \leq \int_0^{2\pi} \text{Tr} \left[(\varphi_n(z)\varphi_{n+\ell}^{-1}(z)[\varphi_{n+\ell}^*(z)]^{-1}\varphi_n^*(z))^{1/2} + I \right]^2 d\theta \\
 & = \int_0^{2\pi} \text{Tr} \left[\varphi_n(z)\varphi_{n+\ell}^{-1}(z)[\varphi_{n+\ell}^*(z)]^{-1}\varphi_n^*(z) \right. \\
 & \quad \left. + 2(\varphi_n(z)\varphi_{n+\ell}^{-1}(z)[\varphi_{n+\ell}^*(z)]^{-1}\varphi_n^*(z))^{1/2} + I \right] d\theta \\
 & = 2\pi p + 2 \int_0^{2\pi} \text{Tr} (\varphi_n(z)\varphi_{n+\ell}^{-1}(z)[\varphi_{n+\ell}^*(z)]^{-1}\varphi_n^*(z))^{1/2} d\theta + 2\pi p,
 \end{aligned}$$

where we used the finite orthogonality (2.8) for the first term. To simplify the notation, we let $A = \varphi_n(z)\varphi_{n+\ell}^{-1}(z)[\varphi_{n+\ell}^*(z)]^{-1}\varphi_n^*(z)$, then we have for the second term

$$\begin{aligned}
 \int_0^{2\pi} \text{Tr} A^{1/2} d\theta &= \int_0^{2\pi} \sum_{i=1}^p \lambda_i^{1/2}(A) d\theta, \\
 &\leq p^{1/2} \int_0^{2\pi} \left[\sum_{i=1}^p \lambda_i(A) \right]^{1/2} d\theta \\
 &\leq (2\pi p)^{1/2} \left(\int_0^{2\pi} \sum_{i=1}^p \lambda_i(A) d\theta \right)^{1/2}
 \end{aligned}$$

where we have used Cauchy-Schwarz for integrals in the last inequality and Cauchy-Schwarz for sums in the inequality before. We therefore find, using the finite orthogonality (2.8),

$$(5.4) \quad \int_0^{2\pi} \text{Tr} (\varphi_n(z)\varphi_{n+\ell}^{-1}(z)[\varphi_{n+\ell}^*(z)]^{-1}\varphi_n^*(z))^{1/2} d\theta \leq 2\pi p.$$

Insert this in (5.3) to find

$$\begin{aligned}
 (5.5) \quad & \int_0^{2\pi} \|\varphi_n(z)\varphi_{n+\ell}^{-1}(z)[\varphi_{n+\ell}^*(z)]^{-1}\varphi_n^*(z) - I\|_2 d\theta \\
 & \leq (8\pi p)^{1/2} \left(\int_0^{2\pi} \|(\varphi_n(z)\varphi_{n+\ell}^{-1}(z)[\varphi_{n+\ell}^*(z)]^{-1}\varphi_n^*(z))^{1/2} - I\|_2^2 d\theta \right)^{1/2}.
 \end{aligned}$$

Our goal will now be to show that the integral on the right tends to 0 as $n \rightarrow \infty$, uniformly for all $\ell \geq 1$. Let again $A = \varphi_n(z)\varphi_{n+\ell}^{-1}(z)[\varphi_{n+\ell}^*(z)]^{-1}\varphi_n^*(z)$, then we have for this integral

$$\begin{aligned}
 \int_0^{2\pi} \|A^{1/2} - I\|_2^2 d\theta &= \int_0^{2\pi} \|A - 2A^{1/2} + I\|_2 d\theta \\
 &\leq \int_0^{2\pi} \text{Tr}(A - 2A^{1/2} + I) d\theta \\
 &\leq 2\pi p - 2 \int_0^{2\pi} \text{Tr} A^{1/2} d\theta + 2\pi p.
 \end{aligned}$$

Since we already have the inequality (5.5), our goal is to show

$$(5.6) \quad \liminf_{n \rightarrow \infty} \inf_{\ell \geq 1} \int_0^{2\pi} \text{Tr} \left(\varphi_n(z) \varphi_{n+\ell}^{-1}(z) [\varphi_{n+\ell}^*(z)]^{-1} \varphi_n^*(z) \right)^{1/2} d\theta \geq 2\pi p.$$

Our second step consists in estimating the integral in (5.6). For this we consider the integral

$$\int_0^{2\pi} \text{Tr} [\varphi_n(z) \rho'(\theta) \varphi_n^*(z)]^{1/2} d\theta = \int_0^{2\pi} \sum_{i=1}^p \sigma_i^{1/2}(\varphi_n(z) \rho'(\theta) \varphi_n^*(z)) d\theta.$$

If we write

$$\varphi_n(z) \rho'(\theta) \varphi_n^*(z) = \varphi_n(z) \varphi_{n+\ell}^{-1}(z) \varphi_{n+\ell}(z) \rho'(\theta) \varphi_n^*(z),$$

and then use (5.1) with $q = 1/2$, $A = \varphi_n(z) \varphi_{n+\ell}^{-1}(z)$, and $B = \varphi_{n+\ell}(z) \rho'(\theta) \varphi_n^*(z)$, then we have

$$\begin{aligned} & \int_0^{2\pi} \text{Tr} [\varphi_n(z) \rho'(\theta) \varphi_n^*(z)]^{1/2} d\theta \\ & \leq \int_0^{2\pi} \sum_{i=1}^p \sigma_i^{1/2}(\varphi_n(z) \varphi_{n+\ell}^{-1}(z)) \sigma_i^{1/2}(\varphi_{n+\ell}(z) \rho'(\theta) \varphi_n^*(z)) d\theta \\ & \leq \int_0^{2\pi} \left(\sum_{i=1}^p \sigma_i(\varphi_n(z) \varphi_{n+\ell}^{-1}(z)) \sum_{i=1}^p \sigma_i(\varphi_{n+\ell}(z) \rho'(\theta) \varphi_n^*(z)) \right)^{1/2} d\theta \\ & \leq \left(\int_0^{2\pi} \text{Tr} \left(\varphi_n(z) \varphi_{n+\ell}^{-1}(z) [\varphi_{n+\ell}^*(z)]^{-1} \varphi_n^*(z) \right)^{1/2} d\theta \right. \\ & \quad \left. \times \int_0^{2\pi} \sum_{i=1}^p \sigma_i(\varphi_{n+\ell}(z) \rho'(\theta) \varphi_n^*(z)) d\theta \right)^{1/2}, \end{aligned}$$

where we have used Cauchy-Schwarz for integrals in the last inequality and Cauchy-Schwarz for sums in the inequality before it. The last integral can be estimated by using (5.1) with $q = 1$, $A = \varphi_{n+\ell}(z) \rho'(\theta)^{1/2}$ and $B = \rho'(\theta)^{1/2} \varphi_n^*(z)$:

$$\begin{aligned} & \int_0^{2\pi} \sum_{i=1}^p \sigma_i(\varphi_{n+\ell}(z) \rho'(\theta) \varphi_n^*(z)) d\theta \\ & \leq \int_0^{2\pi} \sum_{i=1}^p \lambda_i^{1/2}(\varphi_{n+\ell}(z) \rho'(\theta) \varphi_{n+\ell}^*(z)) \lambda_i^{1/2}(\varphi_n(z) \rho'(\theta) \varphi_n^*(z)) d\theta \\ & \leq \int_0^{2\pi} [\text{Tr}(\varphi_{n+\ell}(z) \rho'(\theta) \varphi_{n+\ell}^*(z)) \text{Tr}(\varphi_n(z) \rho'(\theta) \varphi_n^*(z))]^{1/2} d\theta \\ & \leq \left(\int_0^{2\pi} \text{Tr}(\varphi_{n+\ell}(z) \rho'(\theta) \varphi_{n+\ell}^*(z)) d\theta \right. \\ & \quad \left. \times \int_0^{2\pi} \text{Tr}(\varphi_n(z) \rho'(\theta) \varphi_n^*(z)) d\theta \right)^{1/2}. \end{aligned}$$

If we use the finite orthogonality (2.8), then, this gives

$$\int_0^{2\pi} \sum_{i=1}^p \sigma_i(\varphi_{n+\ell}(z) \rho'(\theta) \varphi_n^*(z)) d\theta \leq 2\pi p.$$

Hence we get

$$\begin{aligned} 2\pi p \int_0^{2\pi} \text{Tr}(\varphi_n(z) \varphi_{n+\ell}^{-1}(z) [\varphi_{n+\ell}^*(z)]^{-1} \varphi_n^*(z))^{1/2} d\theta \\ \geq \left(\int_0^{2\pi} \text{Tr}[\varphi_n(z) \rho'(\theta) \varphi_n^*(z)]^{1/2} d\theta \right)^2. \end{aligned}$$

Observe that the right hand side does not depend on ℓ , hence

$$\begin{aligned} (5.7) \quad \inf_{\ell \geq 1} 2\pi p \int_0^{2\pi} \text{Tr}(\varphi_n(z) \varphi_{n+\ell}^{-1}(z) [\varphi_{n+\ell}^*(z)]^{-1} \varphi_n^*(z))^{1/2} d\theta \\ \geq \left(\int_0^{2\pi} \text{Tr}[\varphi_n(z) \rho'(\theta) \varphi_n^*(z)]^{1/2} d\theta \right)^2. \end{aligned}$$

The third step is to estimate the integral on the right. For this we consider the integral

$$\int_0^{2\pi} \text{Tr}(f(\theta) \rho'(\theta) f^*(\theta))^{1/4} d\theta = \int_0^{2\pi} \sum_{i=1}^p \sigma_i^{1/2}(f(\theta) \rho'(\theta)^{1/2}) d\theta,$$

where f is a suitable function. Use (5.1) with $q = 1/2$, $A = f(\theta) \varphi_n(z)^{-1}$ and $B = \varphi_n(z) \rho'(\theta)^{1/2}$, then

$$\begin{aligned} \int_0^{2\pi} \text{Tr}(f \rho' f^*)^{1/4} d\theta &\leq \int_0^{2\pi} \sum_{i=1}^p \sigma^{1/2}(f \varphi_n^{-1}) \sigma^{1/2}(\varphi_n \rho'(\theta)^{1/2}) d\theta \\ &\leq \int_0^{2\pi} \left(\sum_{i=1}^p \sigma(f \varphi_n^{-1}) \sum_{i=1}^p \sigma(\varphi_n \rho'(\theta)^{1/2}) \right)^{1/2} d\theta \\ &\leq \left(\int_0^{2\pi} \sum_{i=1}^p \sigma(f \varphi_n^{-1}) d\theta \int_0^{2\pi} \sum_{i=1}^p \sigma(\varphi_n \rho'(\theta)^{1/2}) d\theta \right)^{1/2} \\ &\leq \left(p \int_0^{2\pi} \sum_{i=1}^p \sigma^2(f \varphi_n^{-1}) d\theta \right)^{1/4} \left(\int_0^{2\pi} \text{Tr}(\varphi_n \rho' \varphi_n^*)^{1/2} d\theta \right)^{1/2} \\ &= \left(2\pi p \frac{1}{2\pi} \int_0^{2\pi} \text{Tr}(f \varphi_n^{-1} (\varphi_n^*)^{-1} f^*) d\theta \right)^{1/4} \\ &\quad \times \left(\int_0^{2\pi} \text{Tr}(\varphi_n \rho' \varphi_n^*)^{1/2} d\theta \right)^{1/2}, \end{aligned}$$

where we use the Cauchy-Schwarz inequality for sums twice and for integrals once. if we take the limit as $n \rightarrow \infty$, then

$$(5.8) \quad \int_0^{2\pi} \text{Tr}(f \rho' f^*)^{1/4} d\theta \\ \leq \left(2\pi p \text{Tr} \int_0^{2\pi} f d\rho f^* \right)^{1/4} \liminf_{n \rightarrow \infty} \left(\int_0^{2\pi} \text{Tr}(\varphi_n \rho' \varphi_n^*)^{1/2} d\theta \right)^{1/2}.$$

The first integral on the right is

$$\int_0^{2\pi} f d\rho f^* = \int_0^{2\pi} f \rho' f^* d\theta + \int_0^{2\pi} f d\rho_s f^*,$$

where ρ_s is the singular part of the measure. If f is a matrix function such that $ff^* \leq cI$, with $c > 0$, then

$$\text{Tr} \int_0^{2\pi} f G_n d\rho_s G_n^* f^* \leq c \text{Tr} \int_0^{2\pi} G_n d\rho_s G_n^*,$$

and this converges to 0. Therefore we replace f in (5.8) by $f G_m$ and let $m \rightarrow \infty$ to find

$$(5.9) \quad \int_0^{2\pi} \text{Tr}(f \rho' f^*)^{1/4} d\theta \\ \leq \left(2\pi p \text{Tr} \int_0^{2\pi} f \rho' f^* \right)^{1/4} \liminf_{n \rightarrow \infty} \left(\int_0^{2\pi} \text{Tr}(\varphi_n \rho' \varphi_n^*)^{1/2} d\theta \right)^{1/2}.$$

Now consider the Schur decomposition of ρ'

$$\rho'(\theta) = U \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_p \end{pmatrix} U^*,$$

where U is a unitary matrix function. The hypothesis that $\det \rho' > 0$ almost everywhere implies that each eigenvalue $\lambda_i > 0$ almost everywhere. By Lusin's theorem there is a sequence of nonnegative continuous functions $f_{n,i}$ with $0 \leq f_{n,i}(\theta) \leq 1/\epsilon$, such that

$$\lim_{n \rightarrow \infty} f_{n,i}(\theta) = \frac{1}{\lambda_i(\theta) + \epsilon}, \quad \text{in measure.}$$

If we take

$$F_m = \begin{pmatrix} f_{m,1}^{1/2} & & & \\ & f_{m,2}^{1/2} & & \\ & & \ddots & \\ & & & f_{m,p}^{1/2} \end{pmatrix} U^*,$$

then $F_m F_m^* \leq e^{-1} I$ and

$$\lim_{m \rightarrow \infty} F_m(\theta) \rho'(\theta) F_m^*(\theta) = \begin{pmatrix} \frac{\lambda_1}{\lambda_1 + \epsilon} & & \\ & \ddots & \\ & & \frac{\lambda_p}{\lambda_p + \epsilon} \end{pmatrix} := \Lambda_\epsilon \quad \text{in measure,}$$

hence replacing f by F_m in (5.9) and using dominated convergence (for convergence in measure) gives

$$(5.10) \quad \int_0^{2\pi} \text{Tr} \Lambda_\epsilon^{1/4} d\theta \leq \left(2\pi p \int_0^{2\pi} \text{Tr} \lambda_\epsilon d\theta \right)^{1/4} \liminf_{n \rightarrow \infty} \left(\int_0^{2\pi} \text{Tr}(\varphi_n \rho' \varphi_n^*)^{1/2} d\theta \right)^{1/2}.$$

Finally we let $\epsilon \rightarrow 0$, then

$$\Lambda_\epsilon \rightarrow I \quad \text{almost everywhere,}$$

and by using dominated convergence (for almost everywhere convergence) we have from (5.10)

$$2\pi p \leq (2\pi p)^{1/2} \liminf_{n \rightarrow \infty} \left(\int_0^{2\pi} \text{Tr}(\varphi_n \rho' \varphi_n^*)^{1/2} d\theta \right)^{1/2}$$

which gives the inequality

$$\liminf_{n \rightarrow \infty} \int_0^{2\pi} \text{Tr}(\varphi_n \rho' \varphi_n^*)^{1/2} d\theta \geq 2\pi p.$$

Combine this with (5.7), then the required inequality (5.6) indeed follows. \square

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